



Efficient gradualism in intertemporal portfolios

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Abstract

This paper examines intertemporal portfolio plans under autocorrelation in asset returns, and considers whether these plans conform to the common advice that risky assets be bought gradually and then held in decreasing amounts as the investment horizon approaches. Given elliptical returns, optimal portfolio plans with precommitment must be mean–variance efficient. Then, for ARMA (1, 1) parameterizations with negative autocorrelation, the age effect (gradual diminishing of risky holdings as the horizon approaches) is confirmed, as is dollar-cost averaging (gradual entry into the risky asset) for sufficiently distant horizons. For a numerically analyzed alternative bivariate returns process, only the age effect is confirmed. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Investment advisors often advocate dollar-cost averaging when acquiring risky asset holdings, and increasing conservatism when approaching retirement. Dollar-cost averaging involves buying the risky asset gradually over several periods rather than all at once, as a way of intertemporally diversifying with respect to the purchase price. The alleged rationale for an age effect on optimal risky holdings is that a greater remaining time to the investment horizon allows

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for greater gains from intertemporal diversification in that a bad rate of return in one period may be offset by a good rate in a later period.

Both of these types of intertemporal advice have proved difficult to confirm analytically, although several papers have made progress in this regard. Samuelson (1989a) shows that age effects can arise under non-constant relative risk aversion.¹ Samuelson (1991) showed that the conventionally advised age effect arises when the risky asset has a two-point distribution and returns are negatively serially correlated. Bodie et al. (1992) showed that age effects arise when younger people have more flexibility in responding to portfolio realizations by altering labor supply. See also Samuelson (1989b, 1990) and Jagannathan and Kocherlakota (1996) for discussions of these issues. On the other hand, Rozeff (1994) showed that linear dollar-cost averaging is mean–variance inefficient if returns are independent through time.

The present paper follows Rozeff in considering the issue of mean–variance efficiency, but allows for serial dependence of returns. Existence of serial correlation in rates of return has been carefully documented by Poterba and Summers (1988) and Fama and French (1988) who find that, for yearly observations of stock returns, rates of return are significantly negatively autocorrelated. This phenomenon reflects a process of ‘mean reversion’ (or, more accurately, trend reversion) in stock prices.²

Like the recent papers of Rozeff (1994), Ehrlich and Hamlen (1995), and others, we consider the precommitment (open-loop) case in which the portfolio sequence is locked in advance. This assumption is realistic since the costs of

¹ Specifically, he considers utility functions $U = \log(W - c)$ or $U = (W - c)^\gamma/\gamma$ for constant $c > 0$, which have decreasing relative risk aversion. He shows that, for given current wealth, someone with longer to go until the investment horizon would hold a larger share in the risky asset, under intertemporal independence of returns. However, his model also implies that as an individual moves through time and wealth grows, on average, faster than the riskfree rate of return, the optimal risky share grows. To get the conventional age effect in this regard, one would have to assume increasing relative risk aversion.

² It is important to realize that mean reversion in asset prices is not necessarily associated with market inefficiency. Balvers et al. (1990) and Cecchetti et al. (1990) show for instance that, with rational decision making by representative agents, trend reversion in the business cycle implies mean reversion in asset prices. Further, Chen (1991), Ilmanen (1995), and others show that decreasing relative risk aversion by the representative consumer can lead to mean reversion.

The mean reversion results have been contested. For instance Kim et al. (1991) argue that the finding of mean reversion in stock prices is due mostly to the pre-WWII period. For the post-WWII years the failure to reject a random walk for stock prices bears some similarity to the failure to reject a random walk for GNP in business cycle data. In both cases the short duration of the time series may be responsible for the failure to reject. An additional argument is presented by Richardson and Stock (1989), who argue that correcting for the use of overlapping data weakens the findings of mean reversion.

gathering and assessing information and implementing plan changes exceed the benefits of doing so for many individuals. Further, many of the insights generated here carry over to situations where this is not so, in which case additional complications are introduced through feedback.³

The nature of the time path of the risky investment in mean–variance efficient plans is considered. With a zero riskfree rate and serial independence of returns, the optimal absolute risky investment is constant over time; Rozeff's (1994) demonstration of the inefficiency of linear dollar-cost averaging emerges as a corollary of this result. With returns following an AR(1) process, the optimal absolute risky investment is constant except that it is lower (higher) in the first and last periods if the first-order serial correlation is negative (positive). For the negative first-order serial correlation case this confirms the conventional wisdom of dollar-cost averaging and decreased absolute and relative holdings of the risky asset as retirement approaches. For ARMA (1, 1) parameterizations giving negative autocorrelation of returns at all lags, the age effect obtains, and dollar-cost averaging also obtains if the horizon is sufficiently distant.⁴ For a more complicated bivariate returns process involving positive short-term and negative long-term autocorrelations, for which analytical results cannot be obtained, simulations reveal only a modified age effect.

As we examine mean–variance efficient portfolios, it is worth recalling the assumptions under which all expected utility maximizing portfolios are mean–variance efficient. By Chamberlain (1983) and Owen and Rabinovitch (1983), mean–variance analysis is equivalent to expected utility analysis if and only if returns are jointly elliptically distributed (which permits bounded distributions as well as the joint normal), and by Nelson (1990) all expected utility maximizing portfolios are then mean–variance efficient. Since in our context

³ Balvers and Mitchell (1997) also consider portfolio choice under mean reversion but without precommitment. Tractability considerations stemming from the absence of precommitment limit the analysis to a single specific utility function, displaying constant absolute risk aversion, whereas the present paper obtains results that are independent of the utility function. Unfortunately, an effort to combine the approach of allowing portfolio response to current conditions with the approach here, of seeking efficient conditions applicable to all utility functions, becomes intractable. In particular, applying Kritzmans (1990) linear investment rules to our analysis would result in the loss of ellipticity of portfolio return, a feature crucial to our approach. The approach of this paper and that of Balvers and Mitchell (1997) lead to similar results concerning the age effect: mean reversion in risky asset prices creates a diversification motive for gradual decline of risky holdings as the horizon approaches.

⁴ The empirical results of Poterba and Summers (1988) and others show mean reversion in low-frequency data but positive autocorrelation at short lags in high-frequency data. Since portfolio advice is often geared towards life-long investments, and hence would often involve low-frequency data, our results justifying the conventional dollar-cost averaging and age effects under mean reversion are of practical import.

returns are jointly elliptical, these results from the literature apply here. Note that all that needs to be assumed about the utility function is that it is increasing and concave, so that mean-standard deviation indifference curves are upward sloped and convex.

Investors with different utility functions will choose different mean–variance efficient portfolios. Nevertheless, any property common to all efficient portfolios will be common to the optimal portfolios of all investors. The properties common to all efficient portfolios are found by characterizing all such portfolios as minimizing the variance of final wealth contingent on some given level of expected final wealth, and noting what features of the efficient portfolios are independent of the given expected wealth level. Because we focus on features common to all efficient portfolios, an important strength of our analysis is that, given our assumptions of precommitment and elliptical returns, *the results apply to all upward sloped, concave utility functions*, not just to some specific utility function.

Section 2 analyzes the intertemporal mean–variance efficient frontier that applies under all dynamic processes. Section 3 presents our main result: the various conditions under which dollar-cost averaging and age effects are or are not optimal under an ARMA (1, 1) process generating excess returns. Section 4 numerically considers an alternative, richer bivariate process, and Section 5 concludes.

2. Deriving efficient time paths of the risky investment

In the T -period problem, final wealth W_T at the end of period T is given by

$$W_T = W_0(1 + r_f)^T + \sum_{t=1}^T w_t x_t, \quad w_t \equiv A_t(1 + r_f)^{T-t}, \quad (1)$$

where W_0 is initial wealth inherited from the end of period 0, x_t represents the excess return of the risky asset over the riskless asset, and r_f is the riskless rate of return. A_t represents the absolute amount invested in the risky asset at the beginning of time period t ; w_t is the future value of the wealth invested in the risky asset.

We consider precommitment portfolio strategies. This approach generates substantial tractability gains, since it avoids the complicating feature of ex ante non-ellipticality of final wealth which plagues analyses that allow future risky asset quantities to be adjusted in an a priori unpredictable way. The results obtained through this approach contain new insights which may also apply in the latter case. Precommitment strategies are particularly relevant in situations in which there are substantial costs of acquiring information and

of using that information to devise and implement changes in the portfolio plan.⁵

The precommitment strategy at time 0 consists of jointly choosing the w_t for all t ($t = 1, \dots, T$) as of the beginning of period 1, based on knowledge of W_0 and r_0 . In any period t , wealth W_{t-1} is inherited from the previous period and is invested according to the precommitted strategy. Then, during period t , the riskfree return r_t and the excess return x_t are realized, resulting in end-of-period wealth W_t . Note that if the rates of return on the risky asset in the various periods are jointly elliptically distributed, as we will assume, then by Eq. (1) final wealth is elliptically distributed, so the conditions of Chamberlain (1983), Owen and Rabinovitch (1983), and Nelson (1990) for mean–variance analysis to be equivalent to expected utility analysis are met.

The optimal time path of w_t will clearly depend in part on the time-zero expectation of the risky return path. For example, if x_0 is above its long-run mean value and if the dynamic process is monotonically mean-reverting, then E_0x_t will be a declining function of time; in this case the fact that E_0x_t is highest in the early periods obviously will impart a tendency for the risky asset to be held more in the early periods. Conversely, if x_0 is less than its long-run mean, the risky asset will be relatively unattractive in the early periods. This effect of the expected risky rate of return is not what we wish to investigate here; instead, we wish to focus on portfolio effects arising from the proximity of the beginning or the end of the sequence of investment periods. Thus we assume that the observed values at the end of period 0 of all variables equal their long-run means, implying that the mean risky rate of return conditional on information at time 0 is constant through time, i.e., $E_0x_t = \bar{x}$ for all t .

From Eq. (1) the mean and variance of W_T , conditional on all information about the dynamic returns process available at the end of time 0, are

$$E_0W_T = W_0(1 + r_f)^T + (w' \iota) \bar{x}, \tag{2}$$

$$\sigma_{W_T}^2 = w' V w, \tag{3}$$

where $\bar{x} \equiv E_0x_t > 0$ (independent of τ), $w \equiv [w_1, \dots, w_T]'$ is the vector of future values of successive absolute investments in the risky asset, ι is a $T \times 1$ vector of ones, and V is a $T \times T$ positive-definite conditional covariance matrix with s, τ element $\text{cov}_0(r_s, r_\tau)$. Thus the structure of V depends on the dynamic process generating the returns. The problem now is to find the intertemporal portfolio

⁵ Any time a principal provides advance portfolio instructions to an agent, there are implicitly prohibitive costs of revising the portfolio plan. The presence of substantial information or revision costs is evidenced, for example, by the casual observation that many individuals only infrequently revise their retirement portfolios.

vector w that minimizes $\sigma_{w_T}^2$ subject to $E_0 W_T = \mu$.⁶ The Lagrangian is

$$L = w'Vw - 2\psi(w'l - B), \tag{4}$$

where $B \equiv [\mu - W_0(1 + r_f)^T]/\bar{x}$, and where 2ψ is the Lagrange multiplier. The first-order conditions are the constraint and

$$Vw - \psi l = 0, \tag{5}$$

implying

$$w^* = \psi^* V^{-1} l. \tag{6}$$

ψ^* is found by using Eq. (6) in the constraint:

$$\psi^* = B/(l'V^{-1}l). \tag{7}$$

The denominator in Eq. (7) is positive because V , and hence V^{-1} , are positive definite. Thus $\psi^* > 0$ provided $B > 0$ – i.e., provided $\mu > W_0(1 + r_f)^T$ (i.e. the mean of final wealth is parametrically set above the wealth which could be achieved by a sequence of riskfree portfolios). We limit our focus to values of μ such that $B > 0$, so $\psi^* > 0$.

Eq. (6) shows the important role of the inverse covariance matrix; such a role is routine in mean–variance portfolio analysis. This equation implies that the future value of the optimal absolute risky investment in period t is proportional to the t th row sum of the inverse covariance matrix. An important aspect of what follows will be to analyze the particular form that these row sums take under the dynamic excess returns processes to be specified in what follows. The row sums, and hence the time profile of the optimal portfolio, will depend in a complicated way on the dynamic process specified and its specific parameter values.

It will be convenient in what follows to analyze the time profile of efficient portfolios in terms of s_t , the risky share of baseline wealth. Define baseline wealth at the start of period t as $W_0(1 + r_f)^{t-1}$; this is what wealth would be if all previous portfolios had been invested entirely in the riskfree asset. Then:

$$\begin{aligned} s_t &= \frac{A_t}{W_0(1 + r_f)^{t-1}} = A_t(1 + r_f)^{T-t}W_0^{-1}(1 + r_f)^{1-T} \\ &= w_t W_0^{-1}(1 + r_f)^{1-T}. \end{aligned} \tag{8}$$

This shows that s_t and w_t are proportional to each other so that the nature of the time path of s_t is the same as that of w_t .

⁶The efficient locus for this T -period problem is linear in $(E W_T, \sigma_{w_T})$ -space. By Meyer (1987), Sinn (1983), and Nelson (1990) the indifference curves under an elliptical joint distribution of returns are upward sloped and convex, so we get at most one local maximum of expected utility.

3. Closed-form portfolio paths under an ARMA(1, 1) returns process

In this section we assume that the excess return follows a general ARMA (1, 1) process:

$$x_t = \rho x_{t-1} + (1 - \rho)b - \delta \varepsilon_{t-1} + \varepsilon_t. \tag{9}$$

Here b is a positive constant representing the long-run mean value of the excess rate of return, and we assume $\delta \in [-1, 1]$ and $\rho \in (-1, 1)$. The process $\{\varepsilon_t\}$ is elliptically distributed white noise. For this process first-order autocorrelation has the sign of $\rho - \delta$ while higher-order autocorrelations (of order k) have the sign of $\rho^{k-1}(\rho - \delta)$. Partial autocorrelations have the sign of $\delta^{k-1}(\rho - \delta)$.

For process (9) we obtain the following analytical results for the time profile of efficient portfolio paths:

Theorem. Assume the ARMA (1, 1) process for the excess return as in Eq. (9). Then for the mean–variance efficient precommitment portfolio plan:

- (a) If $\rho = \delta \in (-1, 1)$, s_t is constant over time.
- (b) If $\delta = 0$ and $-1 < \rho < 0$, s_t is constant for periods 2 through $T - 1$, but is lower in periods 1 and T .
- (c) If $\delta = 0$ and $0 < \rho < 1$, s_t is constant for periods 2 through $T - 1$, but is higher in periods 1 and T .
- (d) If $\delta = 1$ and $-1 < \rho < 1$, the path of s_t is characterized by $s_t > s_{t+1}$ for $t = 1, \dots, T - 1$.
- (e) If $0 < \delta < 1$ and $-1 < \rho < \delta$, then for $t < (T + 1)/2$ we have $s_{t-1} \leq s_t$, while for $t \geq (T + 1)/2$ we have $s_t > s_{t+1}$.
- (f) If $0 < \delta < 1$ and $\delta < \rho < 1$, then for $t \leq (T + 1)/2$ we have $s_{t-1} > s_t$, while for $t > (T + 1)/2$ we have $s_t \leq s_{t+1}$.

Proof. Given Eq. (9), the appendix derives the intertemporal covariance matrix V conditional on information at time 0:

$$V_{st}/\sigma_\varepsilon^2 = \begin{cases} \rho^{\tau-s-1}[\rho(V_{ss}/\sigma_\varepsilon^2) - \delta] & \text{for } s < \tau, \\ 1 + (\rho - \delta)^2(1 - \rho^{2(s-1)})/(1 - \rho^2) & \text{for } s = \tau, \\ \rho^{s-\tau-1}[\rho(V_{\tau\tau}/\sigma_\varepsilon^2) - \delta] & \text{for } s > \tau. \end{cases} \tag{10}$$

Defining $z_t = w_t(\sigma_\varepsilon^2/\psi)$, Eq. (5) implies that

$$\sum_{\tau=1}^T (V_{\tau\tau}/\sigma_\varepsilon^2)z_\tau = 1 \quad \text{for } t = 1, \dots, T. \tag{11}$$

Appendix A shows that for the case of $\delta \neq 1$, Eqs. (10) and (11) yield the following solution for z_t , which by definition is proportional to w_t and therefore

to the risky share s_t of baseline wealth:

$$z_t = \frac{1}{(1 - \delta)^2} [(1 - \rho)^2 + (\rho - \delta)(1 - \rho)\delta^{T-t} + \frac{(1 - \delta\rho)(\rho - \delta)}{1 + \delta}\delta^{t-1} + \frac{(\rho - \delta)^2}{1 + \delta}\delta^{2T-t}]. \tag{12}$$

For $\delta = 1$, Appendix A shows that the path of $\{z_t\}$ is characterized by

$$z_t = z_{t+1} + (1 - \rho)^2 t, \tag{13}$$

with $z_T = \rho + (1 - \rho)T$.

The proof of (a) follows from inspection of Eq. (12). For cases (b) and (c), with $\delta = 0$, (12) implies

$$z_t = \begin{cases} (1 - \rho)^2 & \text{for } t = 2, \dots, T - 1, \\ (1 - \rho)^2 + \rho & \text{for } t = 1, \\ (1 - \rho)^2 + \rho(1 - \rho) & \text{for } t = T, \end{cases} \tag{14}$$

where for $t = 1$ and T use has been made of the identity $0^0 \equiv 1$. Cases (b) and (c) are proven by inspection of Eq. (14). Case (d) is proven by inspection of Eq. (13).

For cases (e) and (f), with $0 < \delta < 1$, define the right-hand side of Eq. (12) as a function f of a continuous variable t , so the discrete sequence $\{z_t\}$ comprises a discrete set of values generated by that function. Then we can differentiate the right side of Eq. (12) with respect to t :

$$\begin{aligned} df/dt = & \frac{- (\ln \delta)(\rho - \delta)\delta^{T-t}}{(1 - \delta)^2(1 + \delta)} [(1 + \delta)(1 - \rho) - (1 - \delta\rho)\delta^{2t-1-T} \\ & + (\rho - \delta)\delta^T]. \end{aligned} \tag{15}$$

Since $0 < \delta < 1$ in the cases under consideration, $\ln \delta$ exists and is negative. Thus

$$\text{sgn } df/dt = \text{sgn}(\rho - \delta)[(1 + \delta)(1 - \rho) - (1 - \delta\rho)\delta^{2t-1-T} + (\rho - \delta)\delta^T] \tag{16}$$

or, equivalently,

$$\text{sgn } df/dt = \text{sgn}[(\rho - \delta)^2(\delta^T - 1) - (\rho - \delta)(1 - \delta\rho)(\delta^{2t-1-T} - 1)]. \tag{17}$$

Noting that with $0 < \delta < 1$ expressions of the form δ^η are ≤ 1 , as $\eta \geq 0$, we have $df/dt < 0$ if $(2t - 1 - T) \geq 0$ and $\rho < \delta$; additionally $df/dt < 0$ if $(2t - 1 - T) \leq 0$ and $\rho > \delta$. Otherwise $df/dt \geq 0$. The cases with $df/dt < 0$ imply $z_t > z_{t+1}$ since $z_t = f(t)$ by Eq. (12), and thus imply $w_t > w_{t+1}$ and therefore $s_t > s_{t+1}$; this proves cases (e) and (f). QED.

Note that if, for example, s_t and hence w_t stays constant over time as in part (a) of the theorem so that A_t grows at the rate r_t , the risky share of actual wealth

A_t/W_t on average declines over time since W_t is expected to grow faster than the rate r_f if positive amounts are invested in the risky asset. Therefore the conventional age effect of a declining risky share obtains even with i.i.d. returns.

The results of the theorem are in terms of the risky share s_t of baseline wealth, whereas discussions of dollar-cost averaging are normally in terms of the absolute risky amount A_t . These differ when $r_f \neq 0$. In order to disentangle this effect of nonzero r_f from the effect of autocorrelated returns, in the exposition which follows we will consider the case in which $r_f = 0$. If $r_f > 0$, an upward tilt is imparted on the time path of A_t relative to that of s_t . Also note that Rozeff's (1994) result, that dollar-cost averaging of absolute risky holdings is mean-variance inefficient when returns are intertemporally independent, emerges as a corollary of case (a) when $\rho = 0 = \delta$ and $r_f = 0$.

The theorem shows, for $\rho < \delta$ with $\delta \in [0, 1]$, that the risky asset holdings decline as the horizon approaches, in absolute terms and therefore also (because on average wealth grows over time) relative to wealth. Dollar-cost averaging always obtains if $\rho < \delta = 0$. Now consider dollar-cost averaging under the parameter conditions of case (e): $\delta \in (0, 1)$ and $\rho < \delta$. The theorem indicates that, in contrast to the situation for later t , for $t < (T + 1)/2$ the slope of the time path is ambiguous in general; however, for $T \rightarrow \infty$ one can see from Eq. (12) that $z_2 > z_1$ so that dollar-cost averaging again obtains. Eq. (12) also reveals that a 'turnpike' result holds. Namely, for $T \rightarrow \infty$ the time that the optimal solution, z_t , spends close to the turnpike outcome, $[(1 - \rho)/(1 - \delta)]^2$, also goes to infinity.⁷

Interestingly, if $\delta = 1$, case (d), dollar-cost averaging is never optimal and no turnpike result obtains. On the other hand, consider the case of negative partial autocorrelation at all lags – which is the empirically relevant case for annual data according to the results of Fama and French (1988) and Poterba and Summers (1988). Since partial autocorrelation of order k has the sign of $\delta^{k-1}(\rho - \delta)$, such negative partial autocorrelation requires $0 < \delta$ and $\rho < \delta$; thus by case (e) both dollar-cost averaging (for large enough T) and the age effect obtain.

Why does negative partial autocorrelation in the returns typically imply the inverse U-shape over time for the absolute amount invested? The answer lies in the analogy with a standard static mean-variance portfolio problem: consider the investment opportunity for each year as a different asset. The main difference then is that here the correlation between the assets is clearly specified – assets

⁷To prove this based on Eq. (12) consider three strictly positive, finite constants: α , β , and γ . Define $\omega = \alpha + \beta + \gamma$. Set $\tau_1 = (\alpha/\omega)T$ and $\tau_2 = [(\alpha + \beta)/\omega]T$ and set t_i equal to the integer closest to τ_i . Now let $T \rightarrow \infty$. Since $T - \tau_i \rightarrow \infty$ and $\tau_i \rightarrow \infty$ it must also be true that $T - t_i \rightarrow \infty$ and that $t_i \rightarrow \infty$. Thus, $z(t_i) \rightarrow [(1 - \rho)/(1 - \delta)]^2$ for $i = 1, 2$. But this must also be true for all t such that $t_1 < t < t_2$. As $\tau_2 - \tau_1 = (\beta/\omega)T \rightarrow \infty$ we have $t_2 - t_1 \rightarrow \infty$, so that the turnpike outcome $[(1 - \rho)/(1 - \delta)]^2$ applies for infinitely many periods as $T \rightarrow \infty$.

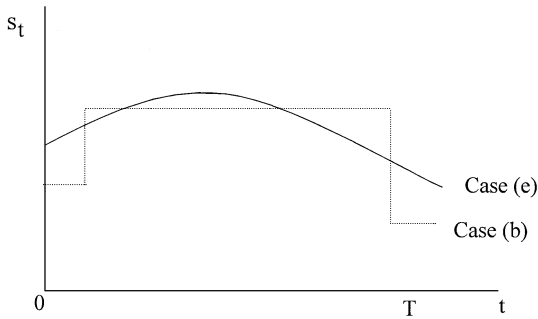


Fig. 1. Risky investment over time with negative autocorrelation.

closer together in time have stronger negative partial autocorrelation. It is the *partial* autocorrelation which determines the marginal risk impact (that is, the contribution to overall portfolio risk) of such a time-specific asset. Accordingly, assets in the early investment periods have fewer assets that are strongly negatively partially correlated with them to offset their risk and thus are less popular and the same is true for assets close to the investment horizon. Assets towards the middle of the investment sequence are the most ‘diversified’, and thus are acquired in larger quantity. In contrast, in the case of independent returns as considered by Rozeff (1994) – case (a) – the position of the asset in time is irrelevant, so a flat profile of risky holdings results.

Fig. 1 displays the typical pattern of the risky share of baseline wealth under mean reversion (negatively autocorrelated returns, which in our context implies negative partial autocorrelation at all lags). Assuming the investment horizon is far enough away, case (e) yields the inverse U-shape as indicated by the bold line. The shape is not symmetric. The reason is that due to the precommitment assumption risk increases over time: the conditional variance of the risky return rises as time is further from the decision period. This by itself would cause the risky share of baseline wealth to fall over time and explains why dollar-cost averaging may not occur, even in case (e), when the investment horizon is near. The dotted line in Fig. 1 displays the AR(1) case, case (b), which has partial autocorrelation that is negative at order one and zero at higher orders. Thus negative partial autocorrelation is only with the nearest asset, which explains the pattern; the increasing conditional return variance again causes asymmetry, with more risky investment in the first period than in the last.

4. A numerical example for a bivariate returns process

The univariate returns process assumed in the previous section does not allow the autocorrelation function to take on different signs at different lags. However,

as pointed out by Lo and Wang (1995), returns in high-frequency data are positively autocorrelated at short lags but negatively autocorrelated at longer lags. Consequently, in this section we consider a process which is capable of generating such a pattern. Specifically, we employ a discrete-time version of the bivariate Ornstein–Uhlenbeck process used by Lo and Wang. While this process will not permit us to obtain analytical results comparable to those in the theorem for the ARMA(1, 1) process, we do present numerical results based on calibrating the parameters of the process to U.S. post-war data.

Clearly the presence of both positive and negative autocorrelations will produce conflicting effects on the direction of change of portfolio shares over time, since negative autocorrelations create a risk-reducing role for time diversification while positive autocorrelations do the reverse. Intuitively, we expect that when many periods remain until the horizon the negative long-lag autocorrelations will predominate and, as discussed previously, a conventional age effect in which risky investment declines over time will occur. However, when the horizon is very near, the positive short-lag autocorrelations will predominate, causing a reverse age effect.

All of the equations in Section 2, culminating in Eq. (8) for the risky share of baseline wealth, still apply. Instead of Eq. (9) of Section 3, we now assume the following bivariate process for deviations p_t of logged stock prices (with reinvested dividends) from trend

$$p_t = \lambda(p_{t-1} - h_{t-1}) + \varepsilon_t, \tag{18}$$

$$h_t = \rho h_{t-1} + \eta_t. \tag{19}$$

Here η and ε are elliptically distributed white noise, and h is a latent variable. Then the excess return is given by

$$x_t = p_t - p_{t-1} + \theta - r_t, \tag{20}$$

where θ is the slope of the logged stock price trend line. Note that the process in Eqs. (18)–(20) is different from, but not a generalization of, the ARMA(1, 1) process in the previous section.

We will use the unconditional autocovariance function to calibrate the stochastic process of Eqs. (18)–(20) to U.S. data. This autocovariance function, with k denoting the lag, is

$$c(0) = \frac{2}{1 + \lambda} \left(\frac{\lambda^2 \sigma_\eta^2}{(1 - \lambda\rho)(1 + \rho)} + \sigma_\varepsilon^2 \right), \tag{21a}$$

$$c(k) = \frac{\lambda^2}{(1 - \rho\lambda)(\lambda - \rho)} \left[\left(\frac{1 - \rho}{1 + \rho} \right) \rho^k - \left(\frac{1 - \lambda}{1 + \lambda} \right) \lambda^k \right] \sigma_\eta^2 - \left(\frac{1 - \lambda}{\lambda(1 + \lambda)} \right) \lambda^k \sigma_\varepsilon^2$$

for $k > 0$. (21b)

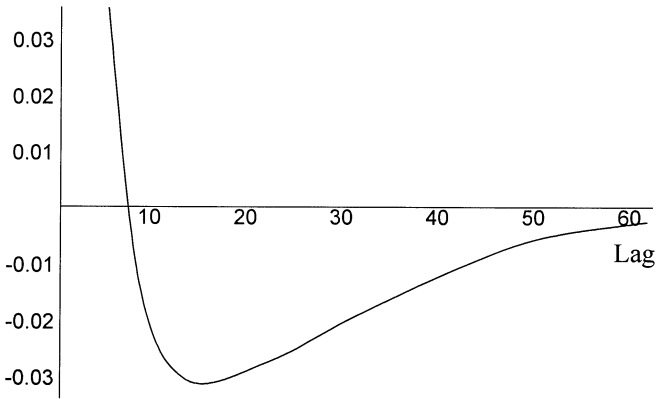


Fig. 2. The unconditional autocorrelation function.

Following Lo and Wang (1995), we perform the calibration by using Eqs. (21a) and (21b) for $k = 0, 1, 5, 25$, but using monthly instead of daily data. Our data consist of monthly excess returns based on the CRSP equal-weighted returns index for 1947.1–1996.12. From these data we compute $c(k)$ for the indicated k -values, and substitute these $c(k)$ values into Eqs. (21a) and (21b) for each of $k = 0, 1, 5, 25$. These values are, respectively, 26.331, 4.134, 0.737, and -0.632 . This substitution yields four equations in the four unknowns $\lambda, \rho, \sigma_\varepsilon^2, \sigma_\eta^2$, which we solve numerically in order to match exactly the four moments. The solution is $\{\lambda, \rho, \sigma_\varepsilon^2, \sigma_\eta^2\} = \{0.946, 0.801, 19.596, 2.927\}$. The resulting autocorrelation function for $k = 0, 1, 2, \dots$, shown in Fig. 2, is positive at short lags and negative at longer lags.

To compute the efficient path of the risky share s_t of baseline wealth, by Eqs. (8) and (6) we need the row sums of the inverse of the conditional autocovariance matrix. Therefore, we use the above numerical parameter values in the following conditional autocovariance function, where we define $c_t(k)$ as the covariance between x_t and x_{t-k} conditional on realizations of all variables through time 0:

$$\begin{aligned}
 c_t(0) = & \frac{\sigma_\eta^2 \lambda^2}{(\lambda - \rho)^2} \left(\frac{(1 - \lambda)}{1 + \lambda} [1 - \lambda^{2(t-1)}] + \frac{(1 - \rho)}{1 + \rho} [1 - \rho^{2(t-1)}] \right. \\
 & \left. - \frac{2(1 - \lambda)(1 - \rho)}{1 - \lambda\rho} [1 - (\lambda\rho)^{t-1}] \right) \\
 & + \frac{\sigma_\varepsilon^2}{1 + \lambda} [2 - (1 - \lambda)\lambda^{2(t-1)}], \tag{22a}
 \end{aligned}$$

$$c_t(k) = \sigma_\eta^2 \left(\sum_{j=k}^{t-1} b_j b_{j-k} \right) - \sigma_\varepsilon^2 \left(\frac{(1 - \lambda)(1 + \lambda^{2(t-k)-1})\lambda^{k-1}}{1 + \lambda} \right) \text{ for } k > 0, \tag{22b}$$

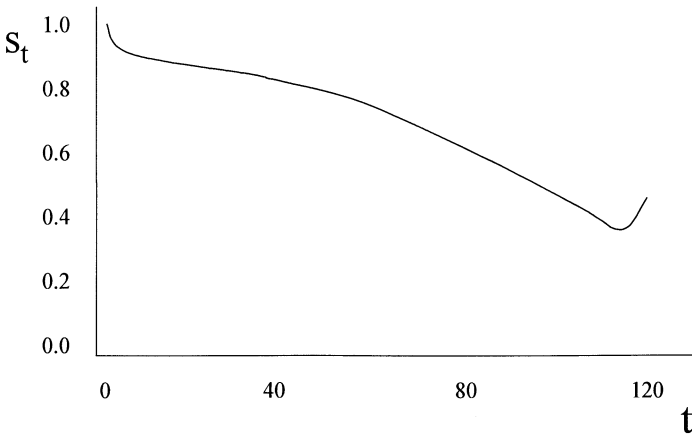


Fig. 3. Normalized risky share of baseline wealth.

where

$$b_j = \frac{\lambda}{\lambda - \rho} [\lambda^{j-1}(1 - \lambda) - \rho^{j-1}(1 - \rho)].$$

Using Eqs. (22a) and (22b) along with Eqs. (6) and (8), we obtain the time path of the risky share s_t of baseline wealth. Since this path depends on the row sums of the $T \times T$ matrix V^{-1} , it depends on the time horizon T . A horizon of five or ten years would be representative for many individuals saving for their children’s college education, for a house purchase, or for retirement. Fig. 3 depicts this path for a ten-year horizon, $T = 120$. Because the magnitude (though not the pattern) of the s_t values depends on the specific location chosen on the efficient frontier, which in turn depends on the utility function, we normalize in Fig. 3 by setting $s_1 = 1$. As our earlier intuition suggested, this time path is generally downward sloped except for an upturn near the horizon; these features are due to the predominance of the negative long-lag autocorrelations when substantial investment time remains, and of the positive short-lag autocorrelations as the horizon gets very near. Hence, the dynamic process of the current section, with the parameter values fitted to U.S. data, demonstrates the conventional age effect with a slight modification.

The qualitative features of the efficient path are the same for the alternative case of a 5 yr horizon, $T = 60$. As indicated these results were based on empirical parameter estimates. To examine sensitivity to the parameter values, for the case of $T = 60$ we performed a grid search, letting λ and ρ range over the interval $(0, 1)$ and letting $\sigma_\varepsilon/\sigma_\eta$ range over $(0, \infty)$. This range for λ and ρ is appropriate as it rules out oscillation of the autocorrelation function and guarantees stationarity of the excess returns process. The grid search revealed

only two qualitative patterns of s_t : that shown in Fig. 3, and a pattern like that in Fig. 3 but without the upturn at the end. Dollar-cost averaging never occurs. To get a feel for why it does not, note that a special case of the bivariate process of this section coincides with a special case of our earlier ARMA(1, 1) process for which the theorem showed that dollar-cost averaging cannot occur. Specifically, when $\sigma_\eta = 0$ the bivariate model collapses to the ARMA(1, 1) model with $\delta = 1$; part (d) of the theorem precluded dollar-cost averaging in this case.

This section has explored the implications of adopting a more complicated process allowing the realistic feature of positive short-lag and negative longer-lag autocorrelations in excess returns. The results reinforce the conventional notion of age effects in long-horizon investment plans.

5. Conclusion

This paper has considered the question of whether mean–variance efficient open-loop portfolio plans in the presence of mean reversion in risky asset prices conform to the commonly given advice that risky assets should be bought gradually and then held in decreasing quantities as the investment horizon approaches. The paper showed the conditions under which mean–variance efficiency is relevant, and demonstrated analytically that relative holdings of the risky asset should indeed be diminished as retirement approaches if returns are ARMA (1, 1) and are negatively correlated at all lags; in this case dollar-cost averaging as the risky asset is first acquired is also optimal, provided the horizon is sufficiently distant. The conventional age effect is also substantially confirmed, numerically, for more complicated dynamics in which returns autocorrelations are positive at short lags and negative at longer lags. These results suggest that investment advisors implicitly predicate their recommendations on the assumption that asset prices are indeed mean reverting.

It should be noted that in practice, when portfolio plans are formulated, the excess return may not be at its long-run average value. If the excess return is initially above its long-run mean, for instance, mean reversion would impart a downward tilt to the time path of risky asset holdings. Thus it can be said that the optimal investment strategy should depend on the phase of the ‘asset return cycle’. An interesting area for future research would be to seek evidence of such a phase-of-cycle effect in investment advice.

Our results were based on the precommitment assumption that individuals lock in their portfolio plans at the beginning of the investment sequence. Such an assumption is reasonable when the collection and assessment of information and the implementation of portfolio revisions are too costly to justify the benefits. While the paper has focused on the precommitment case, the time patterns resulting from this perspective also tend to occur in a closed-loop context, although additional complications are present as well. Further, while

the paper focused on two classes of returns processes, our approach of analyzing the mean–variance efficient locus is valid in general, and can be used to explore the implications of any returns process analytically or numerically. Finally, another potential avenue for future research would be to try to extend the results to the context of n-asset portfolios. Such an extension would be difficult, however, in that it would involve diversification in two dimensions – cross-sectionally and intertemporally – rather than just one.

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Appendix A.

A.1. Derivation of Eq. (10)

To derive Eq. (10), recursively solve for Eq. (9) for realizations between times 0 and t , producing

$$x_t = \rho^t x_0 + (1 - \rho^t)b - \rho^{t-1} \delta \varepsilon_0 + \varepsilon_t + (\rho - \delta) \sum_{i=1}^{t-1} \rho^{i-1} \varepsilon_{t-i}. \tag{A.1}$$

Take conditional expectations in Eq. (A.1) for given information at time 0 and use the definition of covariance, to produce

$$V_{st} = E_0 \left\{ \left[\varepsilon_s + (\rho - \delta) \sum_{i=1}^{s-1} \rho^{i-1} \varepsilon_{s-i} \right] \left[\varepsilon_t + (\rho - \delta) \sum_{i=1}^{t-1} \rho^{i-1} \varepsilon_{t-i} \right] \right\}. \tag{A.2}$$

Now, without loss of generality, assume that $t \geq s$. Keep in mind that the ε_t are serially uncorrelated. Then, for $t > s$,

$$V_{st} = E_0 \left\{ [\varepsilon_s^2 (\rho - \delta) \rho^{t-s-1}] + (\rho - \delta)^2 \left[\sum_{i=1}^{s-1} \rho^{i-1} \varepsilon_{s-i} \right] \left[\sum_{i=t-s+1}^{t-1} \rho^{i-1} \varepsilon_{t-i} \right] \right\}. \tag{A.3}$$

It follows that, for $t > s$,

$$V_{st} = \sigma_\varepsilon^2 \rho^{t-s-1} \left[(\rho - \delta) + (\rho - \delta)^2 \sum_{i=1}^{s-1} \rho^{2(i-1)+1} \right]. \tag{A.4}$$

For $t = s$, it follows from Eq. (A.2) that

$$V_{ss} = \sigma_\varepsilon^2 \left[1 + (\rho - \delta)^2 \sum_{i=1}^{s-1} \rho^{2(i-1)} \right]. \quad (\text{A.5})$$

Since $V_{st} = V_{ts}$, Eqs. (A.4) and (A.5) immediately imply Eq. (10).

A.2. Derivation of Eq. (12)

In Eq. (11), subtract ρ times line t from line $t + 1$ for $t = 1, \dots, T - 1$. This yields, with the help of Eq. (10),

$$1 - \rho = -\delta z_t + [1 - \delta(\rho - \delta)]z_{t+1} + \sum_{i=0}^{T-(t+2)} \rho^i (1 - \delta\rho)(\rho - \delta)z_{t+i+2},$$

$$t = 1, \dots, T - 1. \quad (\text{A.6})$$

For $t = 1$, Eqs. (10) and (11) produce

$$1 = z_1 + \sum_{i=0}^{T-2} \rho^i (\rho - \delta)z_{i+2}. \quad (\text{A.7})$$

Add Eq. (A.6) for $t = 1$ to $[\delta$ times (A.7)], obtaining Eq. (A.8):

$$1 - \rho + \delta = z_2 + \sum_{i=0}^{T-3} \rho^i (\rho - \delta)z_{i+3}. \quad (\text{A.8})$$

Next add Eq. (A.6) for $t = 2$ to $[\delta$ times (A.8)], obtaining Eq. (A.9):

$$\delta(1 - \rho + \delta) + 1 - \rho = z_3 + \sum_{i=0}^{T-4} \rho^i (\rho - \delta)z_{i+4}. \quad (\text{A.9})$$

Denoting the successive right-hand sides of Eqs. (A.7), (A.8), ... as y_t ($t = 1, 2, 3, \dots, T - 1$), continuing this process yields

$$y_t = z_t + \sum_{i=0}^{T-t-1} \rho^i (\rho - \delta)z_{t+i+1}, \quad t = 1, \dots, T - 1 \quad (\text{A.10})$$

where $y_t = \delta y_{t-1} + 1 - \rho$, $y_1 = 1$.

Delaying Eq. (A.10) by one period and adding this to ρ times (A.10) yields $y_{t-1} - \rho y_t = z_{t-1} - \delta z_t$. Applying the recursive formula for y_t produces

$$(1 - \delta\rho)y_t - \rho(1 - \rho) = z_t - \delta z_{t+1}, \quad t = 1, \dots, T - 1. \quad (\text{A.11})$$

To obtain an expression for time T as well, take Eq. (A.6) for $t = T - 1$:

$$1 - \rho = -\delta z_{T-1} + [1 - \delta(\rho - \delta)]z_T. \quad (\text{A.12})$$

To find z_T , add δ times (A.11) evaluated at $T - 1$ to Eq. (A.12) and use $y_T = \delta y_{T-1} + 1 - \rho$ to yield $z_T = y_T$. Consider first the case of $\delta \neq 1$. Then y_T is obtained by solving the difference equation $y_t = \delta y_{t-1} + 1 - \rho$, $y_1 = 1$, producing

$$y_t = \frac{1 - \rho + (\rho - \delta)\delta^{t-1}}{1 - \delta}, \quad t = 1, \dots, T \tag{A.13}$$

and then evaluating at T .

Solving Eq. (A.11) using Eq. (A.13) gives

$$z_t = \delta^{T-t} z_T + \left(\frac{1 - \rho}{1 - \delta} \right)^2 (1 - \delta^{T-t}) + \frac{(1 - \delta\rho)(\rho - \delta)}{(1 - \delta^2)(1 - \delta)} \delta^{t-1} [1 - \delta^{2(T-t)}]. \tag{A.14}$$

Employing the fact that $z_T = y_T$, Eq. (A.13) evaluated at time T and substituted into Eq. (A.14) produces Eq. (12) in the text.

A.3. Derivation of Eq. (13)

As Eqs. (A.13) and (A.14) are not valid for $\delta = 1$, we obtain the $\{z_t\}$ for this case as follows: (A.11) with $\delta = 1$ gives

$$z_{t+1} = z_t + \rho(1 - \rho) - (1 - \rho)x_t. \tag{A.15}$$

The process $y_t = y_{t-1} + (1 - \rho)$ with $y_1 = 1$ has the solution $y_t = \rho + (1 - \rho)t$; using this in Eq. (A.15) gives

$$z_{t+1} = z_t - (1 - \rho)^2 t. \tag{A.16}$$

To obtain the terminal value z_T , we use Eq. (A.15) at $t = T - 1$ to give

$$(\delta - \rho)y_{T-1} - \rho(1 - \rho) - z_{T-1} + z_T = 0 \tag{A.17}$$

and Eq. (A.12) with $\delta = 1$ gives

$$(1 - \rho) + z_{T-1} - (2 - \rho)z_T = 0. \tag{A.18}$$

Adding Eq. (A.17) to Eq. (A.18) and dividing by $(1 - \rho)$ gives

$$y_{T-1} + (1 - \rho) = z_T. \tag{A.19}$$

Since the left side of Eq. (A.19) equals y_T , which equals $\rho + (1 - \rho)T$, we have

$$z_T = \rho + (1 - \rho)T. \tag{A.20}$$

So the solution for the case of $\delta = 1$ is given by Eqs. (A.16) and (A.20), which together constitute Eq. (13) in the text.

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