3B2v8.06a/w (Dec 5 2003).51c XML:ver:5.0.1 DYNCON : 1851 Prod.Type:FTP pp.1–19(col.fig.:NIL) ED:R.JothiL. PAGN:VG SCAN:Geetha

ARTICLE IN PRESS

Available online at www.sciencedirect.com

SCIENCE DIRECT®

ELSEVIER

Journal of Economic Dynamics & Control ▮ (▮▮▮▮) ▮▮▮–▮▮▮

JOURNAL OF Economic Dynamics & Control

www.elsevier.com/locate/jedc

Reducing the dimensionality of linear quadratic control problems

Ronald J. Balvers*, Douglas W. Mitchell

Department of Economics, P.O. Box 6025, West Virginia University, Morgantown, WV 26506-6025, USA

Received 19 July 2003; accepted 1 September 2005

Abstract

In linear-quadratic control (LQC) problems with singularities in the control cost and/or the transition matrices, we derive a reduction of the dimension of the Riccati matrix, simplifying iteration and solution. Employing a novel transformation, we show that, under a certain rank condition, the matrix of optimal feedback coefficients is linear in the reduced Riccati matrix. For a substantive class of problems, our technique permits scalar iteration, leading to simple analytical solution.

JEL classification: C61; C63; D83

Keywords: Linear-quadratic control; Riccati equation; Riccati reduction; Kalman filtering; Intertemporal optimization

1. Introduction

The preeminence of computable general equilibrium models has stimulated interest in the solution procedures for larger-scale models. Most commonly, linear rational expectations models are considered which, typically, are derivable from linear-quadratic control (LQC) problems. The recent work by Sims (2000), Binder

*Corresponding author. Tel.: +304 293 7880; fax: +1 304 293 2233.
E-mail address: rbalvers@wvu.edu (R.J. Balvers).

0165-1889/$ - see front matter
doi:10.1016/j.jedc.2005.09.013

and Pesaran (1997, 2000), Zadrozny (1998), King and Watson (1998), Amman (1997), Amman and Neudecker (1997), Anderson and Moore (1985), Anderson et al. (1997), Ehlgen (1999), and Klein (2000) concentrates on numerical procedures that (1) allow speedy and accurate computation of results, and (2) apply as generally as possible, in particular to systems with noninvertibilities stemming from a singular transition matrix or a singular control cost matrix. These papers improve on the work of Vaughan (1970) and Blanchard and Kahn (1980).[1]

The purpose of this paper is twofold. First, we show that when the control cost matrix or the transition matrix is singular, the dimension of the Riccati equation can be reduced, allowing existing solution techniques or direct iteration for the Riccati equation to become computationally more efficient. Second, we show that for one class of problems explicit analytical solutions for the dynamic (and algebraic) Riccati equations can be obtained, and that for this class, subject to a rank condition, the optimal controls are a linear function of a scalar Riccati kernel.

The paper derives a simple rank expression that places an upper bound on the effective dimensionality of the system for analytical and computational purposes: prior computation of the rank of a composite matrix constructed from all coefficient matrices in the problem statement allows the researcher to establish this bound. The advantage is that one may readily determine up front whether the system has a simple analytical solution, or to what extent reformulation of the problem along the lines delineated here may reduce computation time or improve the transparency of the model.

Duality between LQC and Kalman filtering provides intuition for why the dimensionality of a particular system may be reduced. Consider the following class of Kalman problems.[2] An observation depends linearly on two unobserved state variables following stochastic processes: $w_t = y_{1t} + y_{2t}$. One may describe the uncertainty of the state by considering the conditional variances $\sigma^2_{1t}, \sigma^2_{2t}$ of the state variables and their conditional covariance $\sigma_{12t} = \sigma_{21t}$ (three numbers, stored in a 2×2 covariance matrix). However, conditional on having observed w_t, it is easy to derive from $y_{1t} = w_t - y_{2t}$ that $\sigma^2_{1t} = \sigma^2_{2t} = -\sigma_{12t}$ so that one number is sufficient to describe the state uncertainty. While the intuition for simplification here is straightforward, our rank expression implies a potentially complex interaction between the different singularities in the system that is not always intuitive.

The Kalman application also provides a class of problems for which our reduction approach may provide major computational advantages, namely, for those models in which a Kalman filtering problem is embedded in a larger dynamic model that is not linear quadratic, such as an active learning model: typically, such a model is analyzed with numerical dynamic programming methods in which the 'curse of dimensionality' is a major impediment. Reducing the

[1]With the exception of Binder and Pesaran (2000) these papers focus on infinite horizon problems. The points raised in our paper apply to finite horizon problems as in Binder and Pesaran but are equally relevant for infinite horizon models.

[2]For an example see Claar's (2000, 2005) model of cyclical and natural unemployment rates.

number of state variables in the grid then may provide a substantial computational advantage.[3]

A reduction as proposed here was employed by Balvers and Cosimano (1994) in lowering the dimensionality of their active learning model, but the approach has not been systematically investigated. Mitchell (2000) derived explicit analytical solutions to the 2×1 LQC problem (two target variables and one uncosted control in the control case, or two state variables and one identity in the Kalman case), but his results were not obviously generalizable. In this paper we significantly extend the class of LQC models which can be simplified or even solved analytically.

The paper is organized as follows. Section 2 derives the theorems that state how the dimensionality of the model can be reduced and by how much, and how under a certain rank condition the optimal feedback control matrix is linearly related to the reduced Riccati matrix. The analytical solution is given for one class of cases and examined in Section 3. In Section 4 we conclude the paper by summarizing the advantages of our approach and discussing an algorithm of a MATLAB program (which is further described in Appendix C and is available from the JEDC website at Science Direct) that automates our technique for practical use.

2. Reduction of dimension

2.1. The control problem

In this section we show how to reduce the dimension of the Riccati equation of optimal control. In doing so we illuminate the underlying structure of the dynamics. Two initial lemmas establish the structure of the Riccati matrices, and Theorem 1 gives the reduced dynamics. The reduced problem is shown in Theorem 2 to be sometimes amenable to further simplification of the solution for the control feedback matrix.

The reduction that we present is separate from the concept of reducing a system to 'minimal' form for optimal control or Kalman filtering. A system is in *minimal* form if the number of state variables describing the system cannot be reduced further. This form is attained if and only if each state variable is *controllable* (meaning loosely that the control variables can directly or indirectly impact each state variable) as well as *observable* (meaning loosely that each state variable is relevant in affecting the objective). See Hannan and Deistler (1988). Our Riccati reduction, however, applies even if the system is controllable and observable and, hence, minimal. This reduction can be achieved because the effective dimension of the Riccati matrix (the dimension

[3]The relevant state variables in active learning models include the conditional covariances and (sometimes) the conditional means of the relevant underlying state variables. Reduction of the Riccati dimension does not reduce the number of conditional means but significantly reduces the number of covariances to be considered. As an example, in Wieland (2000) the state space discussed in his Appendix A3 could be reduced from four to two state variables if there were no measurement error in his Eq. (1).

of the Riccati 'kernel') is less than the dimension of the state vector even if the latter is minimal.

2.2. The LQC problem

We start with a general finite horizon LQC problem, Problem 1. The square coefficient matrices $\bar{K}$, $\bar{R}$, and $\bar{A}$ are allowed to be singular and $\bar{J}$ and $\bar{C}$ need not have full row or column rank. *Problem* 1 is

$$V(\bar{y}_s, s) = \underset{\{\bar{u}_t\}_{s+1}^{\mathrm{T}}}{Min}\, E_s\left[\left(\sum_{t=s+1}^{T-1} \beta^t[(\bar{y}_t'\bar{K}\bar{y}_t) + 2(\bar{y}_t'\bar{J}\bar{u}_t) + (\bar{u}_t'\bar{R}\bar{u}_t)]\right) + \beta^T(\bar{y}_T'\bar{K}_T\bar{y}_T + 2\bar{y}_T'\bar{J}_T\bar{u}_T + \bar{u}_T'\bar{R}_T\bar{u}_T)\right], \tag{1a}$$

$$\text{subject to } \bar{y}_t = \bar{A}\bar{y}_{t-1} + \bar{C}\bar{u}_t, \quad t = s+1, \ldots, T, \ \bar{y}_s \textit{ given}, \tag{1b}$$

where the state vector $\bar{y}_t$ is $\bar{n} \times 1$, the control vector $\bar{u}_t$ is $k \times 1$, and random disturbances can be ignored due to certainty equivalence; the coefficient matrices are conformable. The following assumptions are made regarding the system matrices in Problem 1 and are sufficient for second-order conditions to hold:

$$\hat{K} = \begin{pmatrix} \bar{K} & \bar{J} \\ \bar{J}' & \bar{R} \end{pmatrix} \text{ is symmetric } \textit{positive semi-definite},$$

$$(\bar{C}' \ \ I_k)\hat{K}\begin{pmatrix} \bar{C} \\ I_k \end{pmatrix} \text{ is } \textit{positive definite},$$

$$\hat{K}_T = \begin{pmatrix} \bar{K}_T & \bar{J}_T \\ \bar{J}_T' & \bar{R}_T \end{pmatrix} = c\hat{K}, \quad \text{with scalar } c>0.$$

The simple transformation discussed in Appendix A converts LQC Problem 1 into Problem 2. LQC *Problem* 2 is:

$$V(y_s, s) = \underset{\{u_t\}_{s+1}^{\mathrm{T}}}{Min}\left(y_T'K_Ty_T + \sum_{t=s+1}^{T-1} y_t'Ky_t\right) \tag{2a}$$

$$\text{subject to } y_t = Ay_{t-1} + Cu_t, \quad t = s+1, \ldots, T, \quad y_s \text{ given}, \tag{2b}$$

where the state cost matrices K and K_T and the transition matrix A are $n \times n$, the control multiplier matrix C is $n \times k$, the state vector y_t is $n \times 1$, and the control vector u_t is $k \times 1$. The following assumptions are made regarding the system matrices in Problem 2:

- K is *positive definite*,
- C has *full column rank* with state variables ordered so that the submatrix

consisting of its last k rows is invertible,
- $K_T = cK$, with scalar $c>0$.

The transition matrix A need not have full rank. The full column rank of C (no redundant controls) is necessary and sufficient for second-order conditions to hold given that K is positive definite. If K is positive definite the remaining assumptions on Problem 2 are guaranteed to hold based on the assumptions made in Problem 1.

The transformation in Appendix A first increases the size of the state vector by the number of controls, from $\bar{n}$ to $\bar{n}+k$, but standard transformations to render the problem's state space minimal reduce the size of the state vector by d (related in part to the rank deficiency of the control cost matrix $\bar{R}$ and the cross product cost matrix $\bar{J}$) to $n=\bar{n}+k-d$.[4] This process (which we have programmed in MATLAB) initially may or may not decrease the dimension of the state vector. However, the idea described in the following is that starting from a dimension n of the state vector, the dimension of the Riccati matrix in LQC Problem 2 can be reduced to a size of $n-k$ where k is the size of the control vector. This dimension is $n-k=\bar{n}-d\leqslant\bar{n}$ which is strictly lower than the original dimension $\bar{n}$ whenever rank deficiencies related to the control cost/cross product cost matrix exist ($d>0$). For instance, in the large class of LQC problems where the initial problem has no control costs (including no cross product costs) and has an invertible state cost matrix, we have $d\geqslant k$: Problem 2 below applies directly and our following reduction technique reduces the dimension of the Riccati matrix from $n=\bar{n}$ to $n-k$ or less (less if additional singularities exist). In general, if singularities exist in the control cost matrix or the transition matrix, then our technique usually reduces the dimension of the Riccati dynamics, sometimes to the point where an analytical solution is possible.

It is well known (for example, Chow, 1975) that the optimal controls for Problem 2 are given by:

$$u_t^{\text{opt}} = -(C'H_tC)^{-1}C'H_tAy_{t-1} \equiv -F_t y_{t-1}, \quad t\leqslant T, \tag{3}$$

$$H_{t-1} = K + A'H_tA - A'H_tC(C'H_tC)^{-1}C'H_tA, \quad H_T = K_T, \quad t\leqslant T, \tag{4}$$

where the symmetric $n\times n$ matrix H_t is positive definite.[5]

---

[4] If $(\bar{J}'\bar{R})$ does not have full row rank, then the bottom k state variables can be transformed to create a number of rows of zeros, and by symmetry columns of zeros, equal to the rank deficiency. Since the transformation only impacts the zeros in the A matrix (see Appendix A), the A matrix is unchanged. It is then clear that the state variables associated with each row of zeros are not observable because these state variables affect neither the objective nor other state variables. The simplest case is when both control costs and cross product costs are zero, in which case no transformation is necessary to see that (at least) k unobservable state variables can be removed.

[5] Positive definite H_t follows from the assumption that K and K_T are positive definite. We can rewrite Eq. (4) as $H_{t-1}=K+(A-CF_t)'H_t(A-CF_t)$, as can be confirmed by straightforward matrix multiplication, with F_t defined in Eq. (3). Starting with $H_T=K_T$ we can then see that the left-hand side is the sum of a positive definite and a positive semi-definite matrix which must be positive definite. The proof for H_t for arbitrary t follows by induction.

2.3. Some initial intuition

Combining the equations of motion (2b) with the optimal control choice (3) yields the optimally controlled state variables:

$$y_t^{\text{opt}} = [I - C(C'H_tC)^{-1}C'H_t]Ay_{t-1}. \tag{5}$$

Pre-multiplying both sides of Eq. (5) by $C'H_t$ gives that

$$C'H_t y_t^{\text{opt}} = 0_{k,n}. \tag{6}$$

This means that the optimal control choices in each period generate k linear dependencies among the n state variables (remember that C has full column rank k and that H_t has full rank n): in LQC without control costs, associated with each control is a linear dependency; thus, the essential dynamics of the decision problem has dimension $n-k$.

To see more clearly how the dimensionality of the Riccati dynamics is reduced separate from issues related to minimality, our MATLAB program (available from the JEDC website at Science Direct) converts any LQC problem first into minimal form. This is not necessary for our reduction technique to apply but is clearly sensible as it allows a direct reduction in the number of state variables describing the system in Problem 2 that does not affect our basic reduction that we apply subsequently. Moreover, the reduction of the state space to minimality is often sufficient to render the state cost matrix K invertible which is essential for our reduction technique to apply.[6]

How, then, is it possible that a dimension less than the minimal number of state variables is sufficient to describe the Riccati dynamics of the model? The answer, with details provided in the following, is that only $n-k$ state variables are involved in obtaining the value function and capturing the dynamics of what we call the Riccati kernel. The remaining k state variables are necessary only to capture the state dynamics (as implied by minimality).[7] Both the size of the minimal state vector and the dimension of the Riccati kernel are affected by complicated interactions between the system matrices.

2.4. The basic reduction

Eq. (4) can be written as Eqs. (7) and (8):

$$H_{t-1} = K + A'P_tA, \quad t \leqslant T, \tag{7}$$

$$P_t = H_t - H_tC(C'H_tC)^{-1}C'H_t, \quad t \leqslant T. \tag{8}$$

[6] We provide a MATLAB program (available from the JEDC website at Science Direct) that executes the reduction even when K is not invertible, using essentially a generalized inverse. While this program works perfectly in the large set of the examples we have tried, we have not been able to extend our formal proof of the reduction approach to apply without imposing invertibility of K.

[7] In the context of the dual Kalman problem n is the number of conditional means of the state variables, while $n-k$ represents the effective size of the conditional variance–covariance matrix.

The approach in this paper is to exploit restrictions inherent in the P_t matrix to simplify the solution of Problem 2.[8] By Eq. (8):

$$P_t C \equiv \begin{pmatrix} P_{1t} & P_{2t} \\ P_{3t} & P_{4t} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0_{n,k}, \quad t \leqslant T, \tag{9}$$

where, defining $q = n - k$, we have P_{1t} is $q \times q$, P_{2t} is $q \times k$, $P_{3t} = P'_{2t}$, and P_{4t} is $k \times k$; C_1 is $q \times k$, and C_2 is $k \times k$. P_{1t} and P_{4t} are symmetric.

Since C is of full column rank, there is at least one $k \times k$ sub-matrix of C that is invertible. Proper prior arrangement of the y_t vector (and concomitant arrangement of C, A, K, and K_T) has put these k rows together at the bottom of C guaranteeing that C_2 is invertible. We derive:

Lemma 1 (*Reduction to the dynamic core of P_t*). *The $n \times n$ matrix P_t can be written as*

$$P_t = M\Phi_t M', \quad \Phi_t \equiv P_{1t}, \quad t \leqslant T \tag{10}$$

$$M \equiv \begin{bmatrix} I_{n-k} \\ -(C'_2)^{-1} C'_1 \end{bmatrix}, \tag{11}$$

where M is an $n \times q$ matrix, and Φ_t is invertible with dimensions $q \times q$.

Proof. See Appendix B.1. □

It is important to relate the Riccati kernel Φ_t to the solution of LQC Problems 1 and 2 – Eqs. (3) and (4) – in a meaningful way. Lemma 2 provides a useful link.

Lemma 2 (*Relating Φ_t and H_t*). *The $q \times q$ matrix Φ_t in Eq.* (10) *is positive definite and is given by*

$$\Phi_t = (M' H_t^{-1} M)^{-1}, \quad t \leqslant T. \tag{12}$$

Proof. See Appendix B.2. □

Employing Lemmas 1 and 2 we provide the dynamics of Φ_t.

Theorem 1 (*Dynamics of Φ_t^{-1}*). *For all $t \in \{s+1, T\}$ we have*:

$$\Phi_{t-1}^{-1} = B_1 - B'_2(\Phi_t^{-1} + B_3)^{-1} B_2, \quad \Phi_T^{-1} = M' K_T^{-1} M, \tag{13}$$

with $B_t = M'K^{-1}M$, $B_2 = M'AK^{-1}M$, $B_3 = M'AK^{-1}A'M$, and M given by Eq. (11).

Proof. See Appendix B.3. □

The reduced Riccati equation (13) has dimension smaller than that of the original Riccati equation (4).

[8] The symmetric $n \times n$ P_t matrix is not typically employed in dealing with optimal control problems, but in the dual Kalman filtering context has the familiar interpretation of the covariance matrix for the unobserved state variable for the current period conditional on current information (while H_t is the covariance matrix conditional on the previous period's information).

2.5. Nonsingular B_2

The B_i matrices in Theorem 1 are all $q \times q$ and only B_2 is not symmetric. B_1 is positive definite and B_3 is positive semi-definite. By Sylvester's inequality (Eq. (B.1.1)), $B_2 (= M'AK^{-1}M)$ can be of full or less than full rank regardless of whether A has full rank. (But if $rank(A) < q$ then B_2 is certainly singular. Section 3.1 below provides an example.)

The sequence of reduced Riccati matrices obtained in Theorem 1 can be used with Eqs. (3), (7), and (10) to obtain the sequence $\{u_t^{\mathrm{opt}}\}$ of optimal controls. However, given the transformations employed here there is a more convenient way of calculating the optimal controls when B_2 has full rank ($q = n - k < n$), as in this case the feedback matrix can be shown to be linear in Φ_t:

Theorem 2 (*Linear calculation of feedback matrix*). *If $rank(B_2) = q$, then in the optimal control solution $u_t^{\mathrm{opt}} = -F_t y_{t-1}$, the feedback matrix F_t is linear in Φ_t for all $t \leqslant T-1$:*

$$F_t = -WM\Phi_t M'A + WKA, \quad t \leqslant T-1, \tag{14}$$

with $W = (C'C)^{-1}C'K^{-1}[I - A'M(M'K^{-1}A'M)^{-1}M'K^{-1}]$.

Proof. See Appendix B.4. $\square$

Thus computation of the sequence $\{F_t\}$ of control feedback matrices involves first computing $F_T = (C'K_TC)^{-1}C'K_TA$ from Eq. (3) with $H_T = K_T$, next iterating Eq. (13) to get $\{\Phi_t\}$, and then using Eq. (14) to obtain the remainder of the feedback matrix sequence. The value of Theorem 2 is that it allows the feedback matrix sequence to be calculated linearly given $\{\Phi_t\}$ and that it may facilitate comparative statics analysis. Theorems 1 and 2 suggest that our method may enhance computational efficiency, at least in some cases, but this is not necessary and we have not attempted to provide numerical evidence.

2.6. Further reduction when B_2 is singular

The case in which $B_2 = M'AK^{-1}M$ in Eq. (13) is singular allows a further reduction. This reduction is imperfectly related to the singularity of the A matrix: deficiencies in the A matrix make it more likely that B_2 is singular but neither imply nor are implied by a singular B_2 matrix. An extension to Theorem 1 available from the authors shows that the size of the Riccati kernel can be reduced at least to the rank r of B_2 when $r < q$. The theorem is of limited use because there is no counterpart to Theorem 2 in this case and because r is not necessarily the minimum size of the Riccati kernel, since the B_2 counterpart generated by the additional reduction may still be singular, allowing subsequent reduction.

This section has shown how to reduce the size of the Riccati matrix of optimal control, thereby simplifying computation of the Riccati iteration and solution and revealing the underlying structure of the dynamics.

3. Implications

3.1. Effective dimension of the system

By the extension of Theorem 1 proven by the authors, an upper bound on the effective dimension of the system is given by the rank of $B_2 \equiv M'AK^{-1}M$ with M defined in Eq. (11). This bound may be determined in advance – that is, before theoretical appraisal, estimation, numerical analysis, or explicit solution of the model. A general indication of the rank of B_2 is obtained by repeated application of Sylvester's inequality (see Eq. (B.1.1)) to the definition of B_2 given in Theorem 1. Recalling that n represents the dimension of the state vector and k the number of controls, Sylvester's inequality yields:

$$rank(A) - 2k \leqslant rank(B_2) \leqslant min[n-k, rank(A)]. \tag{15}$$

Scalar Riccati dynamics is guaranteed if $n-k$ (the size of B_2) $=1$ (or, of course, if $rank(A)=1$). This case will be discussed in the next sub-section.

Before we discuss the scalar case, we present a simple example to illustrate the bounds implied by Eq. (15). Consider a case with $n=3$, $k=1$, $C'=(0\ 0\ 1)$, and $K=I_3$. The 3×3 matrix A is unrestricted. It is always possible, starting from any like-sized problem with any C, to transform the control and state vectors so that $C'=(0\ 0\ 1)$. Then, $B_2=A_1$, where A_1 is the 2×2 upper left block of A. Consequently, there is an infinitude of A matrices for which any of the following hold: (a) $rank(B_2)=rank(A_1)=n-k=2$, when the two 1×2 rows of A_1 are independent, with $rank(A)$ equaling two or three; (b) $rank(B_2)=rank(A_1)=rank(A)-2k=1$, when A has rank one, two, or three and the two 1×2 rows of A_1 are dependent; and (c) $rank(B_2)=rank(A_1)=rank(A)-2k=0$, when all four elements of A_1 are zero so that A must be singular with either rank one or two.

3.2. Analytical solution when rank $(B_2)\leqslant 1$

When $B_2=M'AK^{-1}$ has rank equal to or less than one, the LQC problem allows scalar-based analytical solution. When the rank of B_2 is equal to *zero* because $B_2=0$, Theorem 1 directly shows that Φ_t does not evolve. When the rank of B_2 is equal to *one*, Theorem 1 applies if $n-k=1$ (so B_2 has full rank)[9]:

Theorem 3 (*Analytic solution when Rank*$(B_2)=1$). *If* $rank(B_2)=1$ *and* $n-k=1$, Φ_t *is scalar with solution given as*

$$\Phi_{t-1} = [1 + B_3\Phi_t]/[B_1 + (B_1B_3 - B_2^2)\Phi_t], \quad t \leqslant T, \tag{16}$$

where Theorem 1 *defines* B_1, B_2, *and* B_3, *which are scalar in this case.*

Proof. Eq. (13) implies that Φ_t in Eq. (16) is scalar. □

[9] If the rank of B_2 is less than full and equals 1 so that $n-k>1$, the extension of Theorem 1 available from the authors applies. The results in Theorem 3 and below then continue to hold if we replace the B_i by their scalar counterparts derived in the further reduction.

Note that in the infinite horizon case when $\Phi_t = \Phi$ for all t Eq. (16) takes a simple quadratic form which is the equation usually solved in simple rational expectations models.

Mitchell (2000) finds the solution to a scalar equation of the form of Eq. (16) as follows. Consider first the case $B_1B_3 - B_2^2 \neq 0$, so that Φ_t evolves nonlinearly (unless $B_2 = 0$). Let $x_t = 1/(c + \Phi_t)$ and hence $\Phi_t = (1 - cx_t)/x_t$, where $c = (B_1 - B_3 + r)/[2(B_1B_3 - B_2^2)]$ and $r = [(B_1 - B_3)^2 + 4(B_1B_3 - B_2^2)]^{1/2}$. Then use $\Phi_t = (1 - cx_t)/x_t$ on both sides of Eq. (16) to obtain a linear equation of evolution for x_t:

$$x_{t-1} = \frac{2(B_1B_3 - B_2^2)}{B_1 + B_3 + r} + \left(\frac{B_1 + B_3 - r}{B_1B_3 + r}\right)x_t, \quad t \leqslant T, \tag{17}$$

with solution

$$x_t = \frac{B_1B_3 - B_2^2}{r} + \left(x_T - \frac{B_1B_3 - B_2^2}{r}\right)\left(\frac{B_1 + B_3 - r}{B_1 + B_3 + r}\right)^{T-t}, \quad t \leqslant T. \tag{18}$$

Then the solution for Φ_t is found by putting Eq. (18) into $\Phi_t = (1 - cx_t)/x_t$.

It is also possible for Eq. (16) to give *linear* evolution of Φ_t. This occurs if and only if $B_1B_3 - B_2^2 = 0$. In this linear case the solution of Eq. (16) for Φ_t is obvious and the eigenvalue is B_3/B_1, which (as Mitchell, 2000 shows) may or may not be less than one in magnitude so the linear case may or may not be stabilizable.

To examine the nature of the scalar dynamics, first derive from Eq. (16):

$$\mathrm{d}\Phi_{t-1}/\mathrm{d}\Phi_t = B_2^2/[B_1 + (B_1B_3 - B_2^2)\Phi_t]^2 \geqslant 0. \tag{19}$$

Eq. (19) allows us to identify three qualitatively distinct cases:

Case 1: $B_2 = 0$. Eq. (16) collapses to $\Phi_{t-1} = 1/B_1$ which is constant. Fig. 1(a) shows the dynamics of Φ_t: the steady state is reached in one iteration.

Case 2: $B_2 \neq 0$ and $B_1B_3 - B_2^2 \neq 0$. Note $B_1B_3 - B_2^2$ cannot be negative: we know $B_3 - B_2B_1^{-1}B_2' = M'AK^{-1}[K - M(M'K^{-1}M)^{-1}M']K^{-1}A'M = M'AC(C'KC)^{-1}C'A'M$, where the last equality follows from substituting Eq. (12) into Eq. (10) and the result into Eq. (8), evaluating the resulting identity at $H_t = K$, subtracting K from both sides, and pre- and post-multiplying both sides by K^{-1}. Hence, $B_3 - B_2B_1^{-1}B_2'$ is positive semi-definite, and so in this scalar case multiplying this expression by the positive scalar B_1 establishes $B_1B_3 - B_2^2 \geqslant 0$. Then in this case 2, Eq. (19) implies that $\mathrm{d}\Phi_{t-1}/\mathrm{d}\Phi_t > 0$ and $\mathrm{d}^2\Phi_{t-1}/\mathrm{d}\Phi_t^2 < 0$; and as $\Phi_t \to \infty$ we have $\mathrm{d}\Phi_{t-1}/\mathrm{d}\Phi_t \to 0$. Thus, the time path is monotonic and convergent as displayed inFig. 1(b).

Case 3: $B_2 \neq 0$ and $B_1B_3 - B_2^2 = 0$. By Eq. (16), $\Phi_{t-1} = (1/B_1) + (B_3/B_1)\Phi_t$ so evolution is linear. This permits the stable case of $B_3 < B_1$ shown in Fig. 1(c) (noting that both $B_1(= M'K^{-1}M)$, and $B_3(= M'AK^{-1}A'M)$ must be nonnegative given positive definite K) as well as the unstable case of $B_3 \geqslant B_1$, also shown in Fig. 1(c).

We have shown here how to solve the case of $n - k = 1$ analytically, which was heretofore done only for the $n = 2$, $k = 1$ case by Mitchell (2000). In addition, we have shown how, due to potential singularities in the transition matrix and its

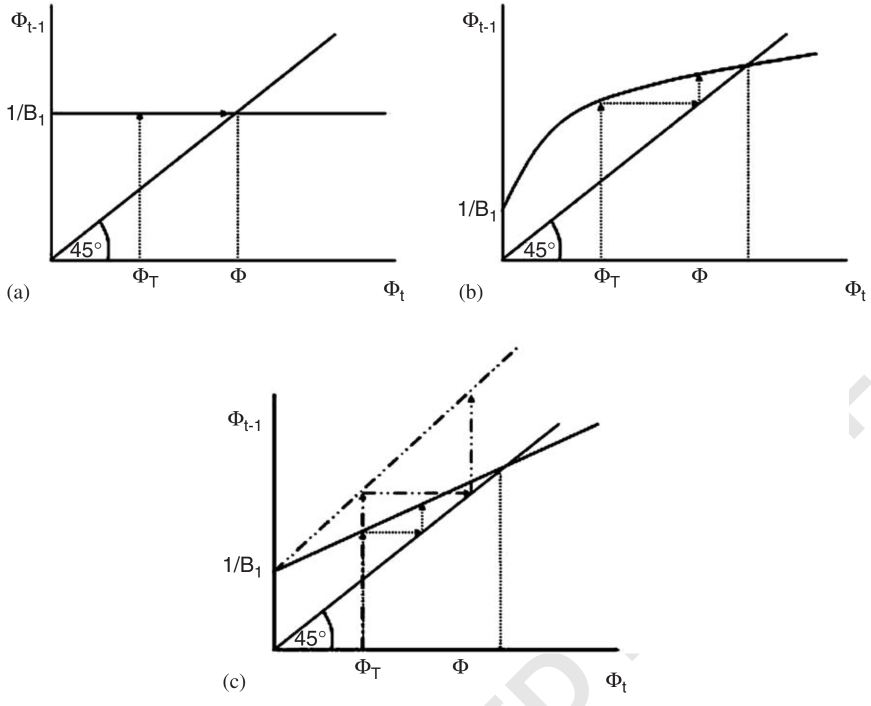

Fig. 1. (a) $B_2 = 0$; (b) $B_2 \neq 0$, $B_1 B_3 > B_2^2$; (c) $B_2 \neq 0$, $B_1 B_3 = B_2^2$.

interactions with the cost matrix, other apparently more complex problems can also be solved analytically if the effective dimensionality equals 1.

3.3. An example

To illustrate some of the advantages of our reduction technique, consider a simple perfect foresight, extended IS/LM model with nominal wage rigidities (see DeLong and Summers, 1986, for a similar model):

$$m_t - p_t = q_t - ai_t \quad \text{(LM)}, \tag{20}$$

$$q_t = b - cr_t \quad \text{(IS)}, \tag{21}$$

$$q_t = \mathrm{d}p_t \quad \text{(aggregate supply)}, \tag{22}$$

$$i_t = r_t + p_{t+1} - p_t \quad \text{(nominal interest rate)}. \tag{23}$$

The government determines the money supply at each time to minimize the value of squared deviations in output from target and squared deviations in inflation from zero:

$$V(s) = \underset{\{m_t\}_{s+1}^{\infty}}{Min} \left(\sum_{t=s+1}^{\infty} [(q_t - q)^2 + h(p_t - p_{t-1})^2] \right). \tag{24}$$

All variables have their standard definitions and are in log terms, and parameters are positive. Simplifying yields the following equations of motion:

$$p_{t+1} = -(b/c) + [(1/a) + (1/c)]q_t + [1 + (1/a)]p_t - (1/a)m_t, \tag{25}$$

$$q_{t+1} = -(db/c) + [(d/a) + (d/c)]q_t + d[1 + (1/a)]p_t - (d/a)m_t. \tag{26}$$

The government objective suggests that we add a constant and a lagged price level variable as state variables. Hence:

$$\bar{y}_t = \begin{pmatrix} 1 \\ p_t \\ p_{t-1} \\ q_t \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -(b/c) & 1 + (1/a) & 0 & (1/c) + (1/a) \\ 0 & 1 & 0 & 0 \\ -(db/c) & d + (d/a) & 0 & (d/c) + (d/a) \end{pmatrix},$$

$$\bar{C} = \begin{pmatrix} 0 \\ -(1/a) \\ 0 \\ -(d/a) \end{pmatrix}, \quad \bar{K} = \begin{pmatrix} q^2 & 0 & 0 & -q \\ 0 & h & -h & 0 \\ 0 & -h & h & 0 \\ -q & 0 & 0 & 1 \end{pmatrix}, \quad \bar{u}_t = m_{t-1}. \tag{27}$$

The problem is not in minimal form since two of the four states are not controllable as can be checked by inspecting the rank of the controllability matrix $[\bar{C}|\bar{A}\,\bar{C}|\bar{A}^2\bar{C}|\bar{A}^3\bar{C}]$ which is two. Based on Rubio (1971, pp. 194–206), transforming the decision problem to minimal form requires finding two independent columns from the controllability matrix and calling this 4×2 matrix S_c. Then find a pseudo-inverse V_c (2×4) so that $V_c S_c = I$. The system obtained as $K = S_c' \bar{K} S_c$, $C = V_c \bar{C}$, $A = V_c \bar{A} S_c$, $y_t = V_c \bar{y}_t$ generates the same optimal controls and loss function as the original problem but based on fewer state variables.

We find

$$S_c = [\bar{C}|\bar{A}\,\bar{C}] = \begin{pmatrix} 0 & 0 \\ 1-\alpha & (1-\alpha)(\alpha + \gamma\delta) \\ 0 & 1-\alpha \\ \delta(1-\alpha) & \delta(1-\alpha)(\alpha + \gamma\delta) \end{pmatrix},$$

and can easily choose some V_c matrix. We pick

$$V_c = \begin{pmatrix} 0 & 1/(1-\alpha) & (\alpha + \gamma\delta)/(1-\alpha) & 0 \\ 0 & 0 & 1/(1-\alpha) & 0 \end{pmatrix},$$

such that $V_c S_c = I$, where we have defined $\alpha = 1 + (1/a)$, $\gamma = (1/c) + (1/a)$, $\delta = d$.

This yields

$$K = \begin{pmatrix} (1-\alpha)^2(\delta^2+h) & (1-\alpha)^2[(\delta^2+h)(\alpha+\gamma\delta)-h] \\ (1-\alpha)^2[(\delta^2+h)(\alpha+\gamma\delta)-h] & (1-\alpha)^2[(\alpha+\gamma\delta)^2(\delta^2+h)+h-2h(\alpha+\gamma\delta)] \end{pmatrix},$$

$$C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 \\ 1 & \alpha+\gamma\delta \end{pmatrix}, \quad y_t = \begin{pmatrix} 1/(1-\alpha) & (\alpha+\gamma\delta)/(1-\alpha) \\ 0 & 1/(1-\alpha) \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \end{pmatrix}. \tag{28}$$

Two state variables can be dropped: in this case the constant and q_t, which from Eq. (22) is tied directly to p_t. In principle, judicious choice of state variables should accomplish a minimal formulation without requiring a transformation, but this is not always easy. The problem now is clearly minimal since K has full rank implying observability and $[C|AC] = I$ and thus has full rank implying controllability.

Since there is one control variable, our basic reduction implies that the dimension of the Riccati kernel governing the dynamics can be reduced from n to $n-k$, that is, from two to one, so that explicit solution is possible. Applying Theorem 1, we first obtain $M' = (0\ \ 1)$. Then we can find

$$B_1 = \frac{1+(\delta^2/h)}{(1-\alpha)^2\delta^2} > 0, \quad B_2 = \frac{1}{(1-\alpha)^2\delta^2} > 0, \quad B_3 = \frac{1}{(1-\alpha)^2\delta^2} > 0. \tag{29}$$

Further, $B_1B_3 - B_2^2 = (1/(1-\alpha)^4\delta^2 h) > 0$. Hence, we know that Case 2 applies as depicted in Fig. 1(b). In addition, since B_2 has full rank, we can apply Theorem 2:

$$F_t = \begin{pmatrix} 1 \\ \alpha+\gamma\delta \end{pmatrix}' [(\alpha - 1 + \gamma\delta) + \Phi_t/h(1-\alpha)^2]. \tag{30}$$

The feedback control policy for both of the state variables is linear in the (scalar) Riccati kernel.

The Riccati kernel can be obtained from Eq. (16) as

$$\Phi_{t-1} = (1-\alpha)^2 h[(1-\alpha)^2\delta^2 + \Phi_t]/[(1-\alpha)^2(\delta^2+h) + \Phi_t]. \tag{31}$$

Since the nonstabilizable case of Fig. 1(c) does not arise in this model, we could solve for the reciprocal of the steady state value Φ of the Riccati kernel:

$$\Phi^{-1} = \{1 + [1 + (4h/\delta^2)]^{1/2}\}/[2(1-\alpha)^2 h]. \tag{32}$$

Note that the full algebraic Riccati matrix is not needed. It can be obtained from Eq. (32) via Eqs. (7) and (10) but is quite complicated.

This example has demonstrated the use of our reduction technique in the context of a relatively simple macroeconomic model which can be expressed in terms of four state variables and one control variable, and has shown how to reduce the Riccati matrix to its scalar kernel and to express the control feedback matrix linearly in terms of the Riccati kernel. Additional examples (including two based on Amman and Neudecker, 1997; Ljungqvist and Sargent, 2000) and a MATLAB program (see

Appendix C) are available from the JEDC website at Science Direct: http://www.sciencedirect.com/science/journal/01651889

4. Summary and conclusion

A procedure has been presented for simplifying and solving LQC models. The procedure is automated in MATLAB and can be summarized in the following algorithm:

Step 1: If necessary, transform the LQC problem to fit the structure of Eqs. (2). Further transformations can, but need not be, employed to make the problem minimal.

Step 2: First obtain M from Eq. (11) and subsequently obtain B_1, B_2, B_3, and Φ_T^{-1} as given in Theorem 1.

Step 3: Substitute $\{\Phi_t\}$ into Eq. (10) to find $\{P_t\}$ and then use Eq. (7) to generate $\{H_t\}$, if $\{P_t\}$ and $\{H_t\}$ are needed. The end matrix H_T is generated as $H_T = K_T$, and H_{T-1} is generated from Φ_T via Eqs. (7) and (10).

Step 4: If B_2 has full rank, find the feedback matrix sequence $\{F_t\}$ from Eq. (3), or, for $t \leqslant T-1$, from Theorem 2. The optimal control vector u_t^{opt} equals $-F_t y_{t-1}$.

Step 4$'$: If B_2 is singular, use $\{H_t\}$ from Step 3 and use Eq. (3) to find the $\{F_t\}$ matrix sequence.

This procedure provides a simple calculation (the rank of B_2) to establish an upper bound on the effective dimension of the problem. It is then possible to find in advance, without computing the solution, how complicated or simple the dynamics and steady state equations are. It is applicable even when the transition matrix and the control cost matrix are singular.

This reduction of the Riccati kernel has several computational and analytical advantages. First, in cases where an LQC problem is embedded in a larger dynamic programming model, such as arises in active learning problems, the curse of dimensionality implies that a reduction in the size of the Riccati kernel may generate substantial computational savings. Second, current techniques for generating numerical solutions to the Riccati equation typically work faster when the dimension of the Riccati dynamics is lower. (Note, however, that the matrices are more compact so the benefits of using any techniques that take advantage of sparsity are reduced.)

Third, it is possible that numerical accuracy is increased because only elementary row and column operations are needed for solution and because most of the operations are imposed on lower-dimension matrices. Patel et al. (1994) point out the numerical advantage of working with smaller-order matrices but also emphasize the numerical instabilities that arise from roundoff errors. Accuracy depends accordingly on condition numbers and on a variety of other factors so is more easily judged on a case-by-case basis. It remains an issue for future research to determine for what class of economic problems the particular matrix inversion required at each iteration involves a sufficiently well-conditioned matrix. Our approach calls for a few initial transformations, involving inverses of matrices that may or may not be ill-conditioned, but subsequently performs iterations on smaller-order matrices.

Fourth, analytical solutions can be obtained if the Riccati kernel is of dimension one (or zero). Such analytical solutions aid economic intuition. Fifth, our approach makes it easier to impose certain numerical restrictions on the coefficient matrices to construct special cases for which the Riccati dimension is one. These solvable cases provide computational advantages by allowing a check on the numerical accuracy of a particular solution algorithm. Sixth, the linearity of the optimal controls in the Riccati kernel may aid in theoretical comparative statics analysis and may be particularly efficient in policy-improvement solution algorithms (see for instance Ljungqvist and Sargent, 2000, p. 56).

Finally, our approach can be applied even when the control cost matrix or the transition matrix is singular. If the control cost matrix, the state cost matrix, and the transition matrices are invertible, the solution technique for LQC problems of Binder and Pesaran (2000) can be compared to ours in that it is applicable to finite horizon models and requires merely elementary row and column operations. Their approach, in terms of computationally demanding operations, requires only the inversion of five matrices (four $n \times n$ and one $k \times k$) to find the optimal control solution for the first period (for all periods if the problem is deterministic). Our approach is more demanding in that it involves inversion of five matrices (two $n \times n$ plus three $k \times k$) as well as $T-1$ additional $n \times n$ inversions (one for each additional period) to obtain the optimal control for the first period. If optimal controls are calculated for all periods, however, our approach requires no additional inverses, but the Binder and Pesaran approach for a stochastic model involves $T-1$ additional $n \times n$ inversions, making it about as demanding as ours. The benefits of our approach come into play when there are singularities in the control cost matrix or the transition matrix. The Binder and Pesaran approach does not apply in these cases, whereas our approach becomes more efficient: it involves a fixed computational cost of inversion of five matrices (two $n \times n$ plus three $k \times k$) plus a variable cost of $T-1$ additional inversions of size merely $(n-k) \times (n-k)$.

5. Uncited references

Anderson and Moore (1979, 1990); Lancaster and Rodman (1995).

Acknowledgements

The authors thank Victor Claar for stimulating their interest in the subject material, and three referees for very careful and helpful advice.

Appendix A. Transformation of LQC Problem 1 to LQC Problem 2

First, augment the state vector with the control vector and redefine variables to suppress the discount factor. We can then write Problem 1 as:

$$V(y_s, s) = \underset{\{u_t\}_{s+1}^{\mathrm{T}}}{Min} \left(y_T' K_T y_T + \sum_{t=s+1}^{T-1} y_t' K y_t \right).$$

$$\text{subject to } y_t = Ay_{t-1} + Cu_t, \quad t = s+1, \ldots, T.$$

where

$$y_t = \beta^{t/2} \begin{pmatrix} \bar{y}_t \\ \bar{u}_t \end{pmatrix}, \quad u_t = \beta^{t/2} \bar{u}_t, \quad K = \begin{pmatrix} \bar{K} & \bar{J} \\ \bar{J}' & \bar{R} \end{pmatrix}, \quad K_T = cK,$$

$$A = \beta^{1/2} \begin{pmatrix} \bar{A} & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \beta^{1/2} \begin{pmatrix} \bar{C} \\ I \end{pmatrix}.$$

The SOC assumptions of Problem 1 guarantee that K is positive semi-definite and that $C'KC$ has full rank. Clearly C has full column rank. If C_2 (the k bottom rows of C) is not invertible it can always be made so by simple rearrangement of the state variables in y_t. It is easy to check that $K_T = cK$. Hence, with the exception of the requirement that K be positive definite, the assumptions for Problem 2 are satisfied automatically if the problem is derived from Problem 1.

To reduce the dimensionality, the problem is first further transformed into minimal form, using standard transformations (not shown here but executed in our MATLAB programs), before our reduction technique is applied.

Appendix B

B.1. Proof of Lemma 1

From Eq. (9) it is straightforward to relate $P_{3t}(= P_{2t}')$ and P_{4t} to P_{1t}. The first equation in (9) gives $P_{2t} = -P_{1t}C_1C_2^{-1}$. Transpose (to produce P_{3t}) and substitute into the second equation (noting the symmetry of P_{1t} as follows from the symmetry of P_t) which yields $P_{4t} = (C_2')^{-1}C_1'P_{1t}C_1C_2^{-1}$. Then factor out the M and M' matrices to produce Eq. (10). To show that Φ_t is invertible, note from Eq. (8) that P_t can be written as the product $H_t[I_n - C(C'H_tC)^{-1}C'H_t]$, where the matrix in brackets is idempotent with trace equal to $trace(I_n) - trace(I_k)$ and thus rank $n - k = q$. Hence, since H_t has full rank n (see footnote 5 for the proof), P_t has rank q by Sylvester's inequality:

$$rank(X_1) + rank(X_2) - n \leqslant rank(X_1X_2) \leqslant min[rank(X_1), rank(X_2)], \tag{B.1.1}$$

where n is the number of rows in X_2.

Eq. (10) then implies that $rank(\Phi_t) \geqslant q$, and since Φ_t has dimension q it must have full rank.

B.2. Proof of Lemma 2

Post-multiply Eq. (8) by $H_t^{-1}P_t$ and then use the transpose of Eq. (9). This yields $P_t = P_t H_t^{-1} P_t$. Next use Eq. (10) in the right-hand side of this equation and premultiply by $(I_q\ 0)$ and post-multiply by $(I_q\ 0)'$ to pick out the upper left block $P_{1t} \equiv \Phi_t$ of the matrix, yielding:

$$\Phi_t = (I_q\ 0) M \Phi_t M' H_t^{-1} M \Phi_t M' \begin{pmatrix} I_q \\ 0 \end{pmatrix}, \quad t \leqslant T. \tag{B.2.1}$$

Now consider that $(I_q\ 0)M = I_q$, and post-multiply Eq. (B.2.1) by $\Phi_t^{-1}(M' H_t^{-1} M)^{-1}$, to obtain Eq. (12). Positive definiteness follows directly from Eq. (12) given that H_t is positive definite.

B.3. Proof of Theorem 1

Substitute $P_t = M\Phi_t M'$ from Lemma 1 into Eq. (7):

$$H_{t-1} = K + A' M \Phi_t M' A, \quad t \leqslant T. \tag{B.3.1}$$

A standard inversion identity (used later on further occasions) states that given the matrices X_1, X_2, X_3, and X_4, with X_1 and X_4 invertible, we have (Söderström, 1994, pp. 156–157):

$$(X_1 + X_2 X_4^{-1} X_3)^{-1} = X_1^{-1} - X_1^{-1} X_2 (X_4 + X_3 X_1^{-1} X_2)^{-1} X_3 X_1^{-1}. \tag{B.3.2}$$

Applying the identity to (B.3.1) gives:

$$H_{t-1}^{-1} = K^{-1} - K^{-1} A' M (\Phi_t^{-1} + M' A K^{-1} A' M)^{-1} M' A K^{-1}. \tag{B.3.3}$$

Post-multiplying by M and pre-multiplying by M' yields after applying Lemma 2:

$$\Phi_{t-1}^{-1} = M' K^{-1} M - M' K^{-1} A' M (\Phi_t^{-1} + M' A K^{-1} A' M)^{-1} M' A K^{-1} M, \tag{B.3.4}$$

which is Eq. (13). $\Phi_T^{-1} = M' K_T^{-1} M$ follows from Eq. (12) using the fact that $H_T = K_T$ from (4).

B.4. Proof of Theorem 2

From Eqs. (3) and (8) we obtain

$$CF_t = (I_n - H_t^{-1} P_t) A = (I_n - H_t^{-1} M \Phi_t M') A, \quad t \leqslant T. \tag{B.4.1}$$

where the second equality follows from Lemma 1. To obtain the term $H_t^{-1}M$ appearing on the right-hand side of Eq. (B.4.1), we first use Eq. (B.3.3) and the definitions in Theorem 1:

$$H_{t-1}^{-1} M = K^{-1} M - K^{-1} A' M (\Phi_t^{-1} + B_3)^{-1} B_2, \quad t \leqslant T. \tag{B.4.2}$$

Use the solution of Eq. (13) for $(\Phi_t^{-1} + B_3)^{-1}$ in Eq. (B.4.2):

$$H_{t-1}^{-1}M = K^{-1}M - (K^{-1}A'M)B_2^{-1'}(B_1 - \Phi_{t-1}^{-1}), \quad t \leqslant T. \tag{B.4.3}$$

Update Eq. (B.4.3) by one period (making it valid for $t \leqslant T-1$) and substitute into the right side of (B.4.1). Pre-multiplying the left and right sides of Eq. (B.4.1) by $(C'C)^{-1}C'$ yields Eq. (14).

Appendix C. MATLAB program and examples

The programs and examples are available from the JEDC website at Science Direct: http://www.sciencedirect.com/science/journal/01651889

A MATLAB program `RedMain.m` executes the reduction approach in this paper and compares the outcome to that from standard iteration. It first converts any initial problem to minimal order. If the state cost matrix is singular when the problem is minimal it allows the user to call `RedK.m` which performs the reduction even in this case. This program effectively employs generalized inverses and it has worked appropriately for every example we have tried, but we have no formal proof for why it works. If the state cost matrix is positive definite and the B_2 matrix is of full rank the basic reduction approach in the paper is applied, employing Theorems 1 and 2. For singular B_2 matrix the user has the choice to perform the reduction without using Theorem 2 or to use the extension of Theorem 1 available from the authors, which leads to further reduction of the size of the Riccati kernel. Various examples are available (four examples developed for this paper, two examples from Amman and Neudecker, 1997, and two examples from Ljungqvist and Sargent, 2000).

Appendix D

The online version of this article contains additional supplementary data. Please visit doi:10.1016/j.jedc.2005.09.013

References

Amman, H.M., 1997. Numerical methods for linear-quadratic models. In: Amman, H.M., Kendrick, D.A., Rust, J. (Eds.), Handbook of Computational Economics. North-Holland, Amsterdam.

Amman, H.M., Neudecker, H., 1997. Numerical solutions of the algebraic matrix Riccati equation. Journal of Economic Dynamics and Control 21, 363–369.

Anderson, B.D.A., Moore, J.B., 1979. Optimal Filtering. Prentice-Hall, Englewood Cliffs.

Anderson, B.D.A., Moore, J.B., 1990. Optimal Control: Linear Quadratic Methods. Prentice-Hall, Englewood Cliffs.

Anderson, E., Hansen, L.P., McGrattan, E.R., Sargent, T., 1997. Mechanics of forming and estimating dynamical linear economies. In: Amman, H.M., Kendrick, D.A., Rust, J. (Eds.), Handbook of Computational Economics. North-Holland, Amsterdam.

Anderson, G., Moore, G., 1985. A linear algebraic procedure for solving linear perfect foresight models. Economics Letters 17, 247–252.

Balvers, R.J., Cosimano, T.F., 1994. Inflation variability and gradualist monetary policy. Review of Economic Studies 61, 721–738.

Binder, M., Pesaran, M.H., 1997. Multivariate linear rational expectations models: characterization of the nature of the solutions and their fully recursive computation. Econometric Theory 13 (6), 877–888.

Binder, M., Pesaran, M.H., 2000. Solution of finite-horizon multivariate linear rational expectations models and sparse linear systems. Journal of Economic Dynamics and Control 24, 325–346.

Blanchard, O.J., Kahn, C.M., 1980. The solution of linear difference models under rational expectations. Econometrica 48, 1305–1311.

Chow, G.C., 1975. Analysis and Control of Dynamic Economic Systems. Wiley, New York.

Claar, V.V., 2000. The natural rate of unemployment, its estimation, and evaluation of its estimators. Ph.D. Dissertation, West Virginia University.

Claar, V.V., 2005. Is the NAIRU more useful in forecasting inflation than the natural rate of unemployment? Applied Economics, forthcoming.

DeLong, J.B., Summers, L., 1986. Is increased price flexibility stabilizing? American Economic Review, 1031–1044.

Ehlgen, J., 1999. A nonrecursive solution method for the linear-quadratic optimal control problem with a singular transition matrix. Computational Economics 13, 17–23.

Hannan, E.J., Deistler, M., 1988. The Statistical Theory of Linear Systems. Wiley, New York.

King, R.G., Watson, M., 1998. The solution of singular linear-difference systems under rational expectations. International Economic Review 39, 1015–1027.

Klein, P., 2000. Using the generalized Schur form to solve a multivariate linear rational expectations model. Journal of Economic Dynamics and Control 24, 1405–1424.

Lancaster, P., Rodman, L., 1995. Algebraic Riccati Equations. Clarendon Press, Oxford.

Ljungqvist, L., Sargent, T.J., 2000. Recursive Macroeconomic Theory. MIT Press, Cambridge.

Mitchell, D.W., 2000. An analytic Riccati solution for two-target discrete-time control. Journal of Economic Dynamics and Control 24, 615–622.

Patel, R.V., Laub, A.J., van Dooren, P.M. (Eds.), 1994. Introduction and survey. In: Numerical Linear Algebra Techniques for Systems and Control. IEEE Press, Piscataway, NJ.

Rubio, J.E., 1971. The Theory of Linear Systems. Academic Press, New York.

Sims, C.S., 2000. Solving linear rational expectations models. Working paper, Yale University.

Söderström, T., 1994. Discrete-time Stochastic Systems: Estimation and Control. Prentice-Hall, New York.

Vaughan, D.R., 1970. A nonrecursive algebraic solution for the discrete Riccati equation. IEEE Transactions on Automatic Control 15, 597–599.

Wieland, V., 2000. Learning by doing and the value of optimal experimentation. Journal of Economic Dynamics and Control 24, 501–534.

Zadrozny, P.A., 1998. An eigenvalue method of undetermined coefficients for solving linear rational expectations models. Journal of Economic Dynamics and Control 22, 1353–1373.